МАТЕМАТИКА

УДК 512.542

КОНЕЧНЫЕ ГРУППЫ, ВСЕ n-МАКСИМАЛЬНЫЕ (n = 2, 3) ПОДГРУППЫ КОТОРЫХ K-$\mathfrak{U}$-СУБНОРМАЛЬНЫ

В.А. Ковалева[1], **Сяолан Йи**[2]

[1]*Гомельский государственный университет им. Ф. Скорины, Гомель, Беларусь*
[2]*Чжэцзянский Научно-Технический университет, Ханчжоу, Китай*

FINITE GROUPS WITH ALL n-MAXIMAL (n = 2, 3) SUBGROUPS K-$\mathfrak{U}$-SUBNORMAL

V.A. Kovaleva[1], **Xiaolan Yi**[2]

[1]*F. Scorina Gomel State University, Gomel*
[2]*Zhejiang Sci-Tech University, Hangzhou, China*

Приведена полная классификация конечных групп, все n-максимальные (n = 2, 3) подгруппы которых являются K-$\mathfrak{U}$-субнормальными.

Ключевые слова: *n-максимальная подгруппа, K-$\mathfrak{U}$-субнормальная подгруппа, $\mathfrak{U}$-субнормальная подгруппа, сверхразрешимая группа, минимальная несверхразрешимая группа, SDH-группа.*

A full classification of finite groups with all n-maximal (n = 2, 3) subgroups K-$\mathfrak{U}$-subnormal is given.

Keywords: *n-maximal subgroup, K-$\mathfrak{U}$-subnormal subgroup, $\mathfrak{U}$-subnormal subgroup, supersoluble group, minimal nonsupersoluble group, SDH-group.*

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use $\mathfrak{U}$ to denote the class of all supersoluble groups; $G^{\mathfrak{U}}$ denotes the intersection of all normal subgroups N of G with $G/N \in \mathfrak{U}$. The symbol $\pi(G)$ denotes the set of prime divisors of the order of G.

A subgroup H of G is called a 2-*maximal* (*second maximal*) subgroup of G whenever H is a maximal subgroup of some maximal subgroup M of G. Similarly we can define 3-*maximal subgroups*, and so on. If H is n-maximal in G but not n-maximal in any proper subgroup of G, then H is said to be a *strictly* n-maximal subgroup of G.

One of the interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its n-maximal subgroups. The earliest publications in this direction are the articles of L. Rédei [1] and B. Huppert [2]. L. Rédei described the nonsoluble groups with abelian two maximal subgroups. B. Huppert established the supersolubility of G whose all second maximal subgroups are normal. In the same article Huppert proved that if all 3-maximal subgroups of G are normal in G, then the commutator subgroup G' of G is nilpotent and the chief rank of G is at most 2. These results were developed by many authors. In partiqular, L.Ja. Poljakov [3] proved that G is supersoluble if every 2-maximal subgroup of G is permutable with every maximal subgroup of G. He also established the solubility of G in the case when every maximal subgroup of G permutes with every 3-maximal subgroups of G. Some later, R.K. Agrawal [4] proved that G is supersoluble if any 2-maximal subgroup of G is permutable with every Sylow subgroup of G. In [5], Z. Janko described the groups whose 4-maximal subgroups are normal. A description of nonsoluble groups with all 2-maximal subgroups nilpotent was obtained by M. Suzuki [6] and Z. Janko [7]. In [8], T.M. Gagen and Z. Janko gave a description of simple groups whose 3-maximal subgroups are nilpotent. V.A. Belonogov [9] studied those groups in which every 2-maximal subgroup is nilpotent. Continuing this, V.N. Semenchuk [10] obtained a description of soluble groups whose all 2-maximal subgroups are supersoluble. A. Mann [11] studied the structure of the groups whose n-maximal subgroups are subnormal. He proved that if all n-maximal subgroups of a soluble group G are subnormal and $|\pi(G)| \geq n+1$, then G is nilpotent; but if $|\pi(G)| \geq n-1$, then G is φ-dispersive for some ordering φ of the set of all primes. Finally, in the case $|\pi(G)| = n$, Mann described G completely. A.E. Spencer [12] studied groups in which every n-maximal chain contains subnormal subgroup. In partiqular, Spencer proved that G is a Schmidt group with abelian Sylow subgroups if every 2-maximal chain of G contains subnormal subgroup. In [13], M. Asaad studied groups whose strictly

n-maximal subgroups are normal. P. Flavell [14] obtained an upper bound for the number of maximal subgroups containing a strictly 2-maximal subgroup and classify the extremal examples.

Among the recent results on n-maximal subgroups we can mention the paper of X.Y. Guo and K.P. Shum [15]. In this paper the authors proved that G is soluble if all its 2-maximal subgroups enjoy the cover-avoidance property. W. Guo, K.P. Shum, A.N. Skiba and Li Baojun [16,17,18] gave new characterizations of supersoluble groups in terms of 2-maximal subgroups. Li Shirong [19] obtained a classification of nonnilpotent groups whose all 2-maximal subgroups are TI-subgroups. In the paper [20], W. Guo, H.V. Legchekova and A.N. Skiba described the groups whose every 3-maximal subgroup permutes with all maximal subgroups. In [21], W. Guo, Yu.V. Lutsenko and A.N. Skiba gave a description of nonnilpotent groups in which every two 3-maximal subgroups are permutable. Yu.V. Lutsenko and A.N. Skiba [22] obtained a description of the groups whose all 3-maximal subgroups are S-quasinormal. Subsequently, this result was strengthened by Yu.V. Lutsenko and A.N. Skiba in [23] to provide a description of the groups whose all 3-maximal subgroups are subnormal. Developing some of the above-mention results, D.P. Andreeva and A.N. Skiba [24] obtained a description of the groups in which every 3-maximal chain contains a proper S-qausinormal subgroup. Moreover, in [25], W. Guo, D.P. Andreeva and A.N. Skiba obtained the description of the groups in which every 3-maximal chain contains a proper subnormal subgroup. In [26], A. Ballester-Bolinches, L.M. Ezquerro and A.N. Skiba obtained a full classification of the groups in which the second maximal subgroups of the Sylow subgroups cover or avoid the chief factors of some of its chief series. In [27], V.N. Kniahina and V.S. Monakhov studied those groups G in which every n-maximal subgroup permutes with each Schmidt subgroup. In partiqular, it was be proved that if $n = 1, 2, 3,$ then G is metanilpotent; but if $n \geq 4$ and G is soluble, then the nilpotent length of G is at most $n-1$.

Another interesting results on n-maximal subgroups were obtained by V.A. Kovaleva and A.N. Skiba in [28], [29] and V.S. Monakhov and V.N. Kniahina in [30]. In [28], the authors described the groups whose all n-maximal subgroups are $\mathfrak{U}$-subnormal. In [29] a description of the groups with all n-maximal subgroups $\mathfrak{F}$-subnormal for some saturated formation $\mathfrak{F}$ was obtained. In [30], the groups with all 2-maximal subgroups $\mathbb{P}$-subnormal were studied.

Recall that a subgroup H of G is said to be:

(i) $\mathfrak{U}$-*subnormal* in G if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that $H_i / (H_{i-1})_{H_i} \in \mathfrak{U},$ for $i = 1, \ldots, n;$

(ii) $\mathfrak{U}$-*subnormal* (in the sense of Kegel [31]) or K-$\mathfrak{U}$-*subnormal* [32, p. 236] in G if there exists a chain of subgroups

$$H = H_0 \leq H_1 \leq \cdots \leq H_t = G$$

such that either H_{i-1} is normal in H_i or $H_i / (H_{i-1})_{H_i} \in \mathfrak{U}$ for all $i = 1, \ldots, t.$ It is evident that every subnormal subgroup is K-$\mathfrak{U}$-subnormal. The inverse, in general, it is not true. For example, in the group S_3 a subgroup of order 2 is K-$\mathfrak{U}$-subnormal and at the same time it is not subnormal. This elementary observation and the results in [23], [25] make natural the following question:

I. *What is the structure of* G *under the condition that every* 2*-maximal subgroup of* G *is* K-$\mathfrak{U}$*-subnormal?*

II. *What is the structure of* G *under the condition that every* 3*-maximal subgroup of* G *is* K-$\mathfrak{U}$*-subnormal?*

In this paper we given the solutions of these two questions.

1 Preliminary results

The solutions of Question I and Question II are based on the following results.

***Lemma* 1.1.** *Let H and K be subgroups of G such that H is* K-$\mathfrak{U}$*-subnormal in G.*

(1) $H \cap K$ *is* K-$\mathfrak{U}$*-subnormal in K* [32, Lemma 6.1.7 (2)].

(2) *If N is a normal subgroup in G, then* HN/N *is* K-$\mathfrak{U}$*-subnormal in* G/N [32, Lemma 6.1.6 (3)].

(3) *If K is* K-$\mathfrak{U}$*-subnormal in H, then K is* K-$\mathfrak{U}$*-subnormal in G* [32, Lemma 6.1.6 (1)].

(4) *If* $G^{\mathfrak{U}} \leq K$, *then K is* K-$\mathfrak{U}$*-subnormal in G* [32, Lemma 6.1.7 (1)].

The next lemma is evident.

***Lemma* 1.2.** *If G is supersoluble, then every subgroup of G is* K-$\mathfrak{U}$*-subnormal in G.*

***Lemma* 1.3.** *If every n-maximal subgroup of G is* K-$\mathfrak{U}$*-subnormal in* G, *then every* $(n-1)$*-maximal subgroup of* G *is supersoluble and every* $(n+1)$*-maximal subgroup of* G *is* K-$\mathfrak{U}$*-subnormal in* G.

Proof. We first show that every $(n-1)$-maximal subgroup of G is supersoluble. Let H be an $(n-1)$-maximal subgroup of G and K any maximal subgroup of H. Then K is an n-maximal subgroup of G and so, by hypothesis, K is K-$\mathfrak{U}$-subnormal in G. Hence K is K-$\mathfrak{U}$-subnormal in H by Lemma 1.1 (1). Therefore either K is normal in H or $H/K_H \in \mathfrak{U}$. If K is normal in H, then $|H:K|$ is a prime in view of maximality of K in H. Let $H/K_H \in \mathfrak{U}$. Then we also get that

$$|H:K|=|H/K_H:K/K_H|$$
is a prime. Thus H is supersoluble.

Now, let E be an $(n+1)$-maximal subgroup of G, and let E_1 and E_2 be an n-maximal and an $(n-1)$-maximal subgroup of G, respectively, such that $E \le E_1 \le E_2$.

Then, by the above, E_2 is supersoluble, so E_1 is supersoluble. Hence it is easy to see that E is K - $\mathfrak{U}$-subnormal in E_1. By hypothesis, E_1 is K - $\mathfrak{U}$-subnormal in G. Therefore E is K - $\mathfrak{U}$-subnormal in G by Lemma 1.1 (3). The lemma is proved.

***Lemma* 1.4.** *If every* 3*-maximal subgroup of G is K - $\mathfrak{U}$-subnormal in G, then G is soluble.*

Proof. Suppose that lemma is false and let G be a counterexample with $|G|$ minimal. Since every 3-maximal subgroup of G is K - $\mathfrak{U}$-subnormal in G, every 2-maximal subgroup of G is supersoluble by Lemma 1.3. Hence every maximal subgroup of G is either supersoluble or a minimal nonsupersoluble group. Therefore all proper subgroups of G are soluble in view of [2]. Assume that all 3-maximal subgroups of G are identity. Then all 2-maximal subgroups of G have prime orderes and so every maximal subgroup of G is supersoluble. Hence G is either supersoluble or a minimal nonsupersoluble group. Thus in view of [2], G is soluble, a contradiction. Hence there is a 3-maximal subgroup T of G such that $T \ne 1$. Since T is K - $\mathfrak{U}$-subnormal in G, there exists a proper subgroup H of G such that $T \le H$ and either $G/H_G \in \mathfrak{U}$ or H is normal in G. If $G/H_G \in \mathfrak{U}$, then G is soluble in view of solubility of H_G, a contradiction. Therefore H is normal in G. Let E/H be any 3-maximal subgroup of G/H. Then E is a 3-maximal subgroup of G, hence E is K - $\mathfrak{U}$-subnormal in G. Hence E/H is K - $\mathfrak{U}$-subnormal in G/H by Lemma 1.1 (2). Thus the hypothesis holds for G/H. Hence G/H is soluble by the choice of G. Therefore G is soluble. This contradiction completes the proof of the lemma.

2 Description of groups with all 2-maximal or all 3-maximal subgroups K - $\mathfrak{U}$ -subnormal

Recall that G is called *a minimal nonsupersoluble group* provided G does not belong to $\mathfrak{U}$ but every proper subgroup of G belongs to $\mathfrak{U}$. Such groups were described by B. Huppert [2] and K. Doerk [33]. We say that G is a *special Doerk-Huppert group* or an *SDH-group* if G is a minimal nonsupersoluble group such that $G^{\mathfrak{U}}$ is a minimal normal subgroup of G.

The solution of Question I originates to [28], [29], where, in particular, the following theorem was proved.

***Theorem* A*.** *Every* 2*-maximal subgroup of G is $\mathfrak{U}$-subnormal in G if and only if G is either supersoluble or an SDH-group.*

If every 2-maximal subgroup of G is K - $\mathfrak{U}$-subnormal, then every maximal subgroup of G is supersoluble by Lemma 1.3. Therefore in this case G is either supersoluble or a minimal nonsupersoluble group, hence G is soluble by [2]. Thus we get the following

***Theorem* A.** *Every* 2*-maximal subgroup of G is K - $\mathfrak{U}$-subnormal in G if and only if G is either supersoluble or an SDH-group.*

The solution of Question II is more complete. Note that since each subgroup of every supersoluble group is K - $\mathfrak{U}$-subnormal, we need, in fact, only consider the case when G is not supersoluble. But in this case, in view of [28] or [29], $|\pi(G)| \le 4$.

The following theorems are proved.

***Theorem* B.** *Let G be a nonsupersoluble group with $|\pi(G)|=2$. Let p, q be distinct prime divisors of $|G|$ and G_p, G_q be Sylow p-subgroup and q-subgroup of G respectively. Every* 3*-maximal subgroup of G is K - $\mathfrak{U}$-subnormal in G if and only if G is a soluble group of one of the following types:*

I. *G is a minimal nonsupersoluble group and either $|\Phi(G^{\mathfrak{U}})|$ is a prime or $\Phi(G^{\mathfrak{U}})=1$.*

II. *$G=G_p \rtimes G_q$, where G_p is the unique minimal normal subgroup of G and every* 2*-maximal subgroup of G_q is an Abelian group of exponent dividing $p-1$. Moreover, every maximal subgroup of G containing G_p is either supersoluble or an SDH-group and at least one of the maximal subgroup of G is not supersoluble.*

III. *$G=(G_p \times Q_1) \rtimes Q_2$, where $G_q = Q_1 \rtimes Q_2$, G_p and Q_1 are minimal normal subgroups of G, $|Q_1|=q$, $G_p \rtimes Q_2$ is an SDH-group and every maximal subgroup of G containing $G_p \rtimes Q_1$ is supersoluble. Moreover, if $p<q$, then every* 2*-maximal subgroup of G is nilpotent.*

IV. *$G=G_p \rtimes G_q$, where G_p is a minimal normal subgroup of G, $O_q(G) \ne 1$, $\Phi(G) \ne 1$, every maximal subgroup of G containing G_p is either supersoluble or an SDH-group and $G/\Phi(G)$ is a group one of types II or III.*

V. *$G=(P_1 \times P_2) \rtimes G_q$, where $G_p = P_1 \times P_2$, P_1, P_2 are minimal normal subgroups of G, every maximal subgroup of G containing G_p is supersoluble, $P_1 \rtimes G_q$ is an SDH-group and $P_2 \rtimes G_q$ is either an SDH-group or a supersoluble group with $|P_2|=p$.*

VI. $G = G_p \rtimes G_q$, $\Phi(G_p)$ *is a minimal normal subgroup of* G, *every maximal subgroup of* G *containing* G_p *is supersoluble and* $\Phi(G_p) \rtimes G_q$ *is an SDH-group.*

VII. *Every of the subgroups* G_p *and* G_q *is not normal in* G *and the following hold:*

(i) *if* $p < q$, *then* $G = P_1 \rtimes (G_q \rtimes P_2)$, *where* $G_p = P_1 \rtimes P_2$, P_1 *is a minimal normal subgroup of* G, $|P_2| = p$, $G_q = \langle a \rangle$ *is a cyclic group and* $\langle a^q \rangle$ *is normal in* G. *Moreover,* G *has precisely three classes of maximal subgroups whose representatives are* $P_1 \rtimes G_q$, $G_q \rtimes P_2$, $\langle a^q \rangle \rtimes G_p$, *where* $P_1 \rtimes G_q$ *is an SDH-group;*

(ii) *if* $p > q$, *then* $G = P_1(G_q \rtimes P_2)$, *where* $G_p = P_1P_2$, P_1 *is a normal subgroup of* G, $P_2 = \langle b \rangle$ *is a cyclic group and* $1 \neq P_1 \cap P_2 = \langle b^p \rangle$. *Moreover,* G *has precisely three classes of maximal subgroups whose representatives are* $P_1 \rtimes G_q$, $G_q \rtimes P_2$, G_p, *where* $|G : G_q \rtimes P_2| = p$, $P_1 \rtimes G_q$ *is a supersoluble group and* $G_q \rtimes P_2$ *is an SDH-group.*

***Theorem* C.** *Let* G *be a nonsupersoluble group with* $|\pi(G)| = 3$. *Let* p, q, r *be distinct prime divisors of* $|G|$ *and* G_p, G_q, G_r *be Sylow p-subgroup, q-subgroup and r-subgroup of G respectively. Every* 3*-maximal subgroup of G is* K-$\mathfrak{U}$*-subnormal in G if and only if G is a soluble group of one of the following types:*

I. *G is a minimal nonsupersoluble group and either* $|\Phi(G^{\mathfrak{U}})|$ *is a prime or* $\Phi(G^{\mathfrak{U}}) = 1$.

II. $G = G_p \rtimes (G_q \rtimes G_r)$, *where* G_p *is a minimal normal subgroup of* G, *every maximal subgroup of* G *is either supersoluble or an SDH-group and at least one of the maximal subgroups of G is not supersoluble. Moreover, the following hold:*

(i) *if* G_p *is the unique minimal normal subgroup of* G, *then every* 2*-maximal subgroup of* $G_q \rtimes G_r$ *is an Abelian group of exponent dividing* $p-1$;

(ii) *if* $G_q \rtimes G_r$ *is an SDH-group, then every maximal subgroup of* G *containing* G_pG_q *is supersoluble and* $G_p \rtimes G_r$ *is either an SDH-group or a supersoluble group with* $|G_p| = p$.

III. $G = (P_1 \times P_2) \rtimes (G_q \rtimes G_r)$, *where* $G_p = P_1 \times P_2$, P_1, P_2 *are minimal normal subgroups of* G *and* G_q, G_r *are cyclic groups. Moreover, every maximal subgroup of* G *containing* G_p *is supersoluble,* $P_1 \rtimes (G_q \rtimes G_r)$ *is an SDH-group and* $P_2 \rtimes (G_q \rtimes G_r)$ *is either an SDH-group or a supersoluble group with* $|P_2| = p$.

IV. $G = G_p \rtimes (G_q \rtimes G_r)$, $\Phi(G_p)$ *is a minimal normal subgroup of G, every maximal subgroup of G containing* G_p *is supersoluble and* $\Phi(G_p) \rtimes (G_q \rtimes G_r)$ *is an SDH-group.*

***Theorem* D.** *Let* G *be a nonsupersoluble group with* $|\pi(G)| = 4$. *Let* p, q, r, t *be distinct prime divisors of* $|G|$ ($p > q > r > t$) *and* G_p, G_q, G_r, G_t *be Sylow p-subgroup, q-subgroup, r-subgroup and t-subgroup of G respectively. Every* 3*-maximal subgroup of* G *is* K-$\mathfrak{U}$*-subnormal in* G *if and only if* $G = G_p \rtimes (G_q \rtimes (G_r \rtimes G_t))$ *is a soluble group such that* G *has precisely three classes of maximal subgroups whose representatives are* $G_qG_rG_t$, $G_pG_qG_r\Phi(G_t)$, $G_pG_q\Phi(G_r)G_t$ *and* $G_p\Phi(G_q)G_rG_t$, *and every nonsupersoluble maximal subgroup of* G *is an SDH-group,* G_r *and* G_t *are cyclic groups and following hold:*

(1) *if* $G_qG_rG_t$ *is an SDH-group, then* $G^{\mathfrak{U}} = G_p \times G_q$, G_q *is a minimal normal subgroup of* G, *the subgroups* $G_pG_qG_r\Phi(G_t)$ *and* $G_pG_q\Phi(G_r)G_t$ *are supersoluble and* $G_pG_rG_t$ *is either an SDH-group or a supersoluble group with* $|G_p| = p$;

(2) *if* $G_qG_rG_t$ *is a soluble group, then* G_q *is cyclic.*

The classes of groups which are described in Theorems B and C are pairwise disjoint. It is easy to construct examples to show that all classes of the groups in this theorems and in Theorems A and D are not empty. Note also that Theorems B, C and D show that the class of the groups with all 3-maximal subgroups K-$\mathfrak{U}$-subnormal is essentially wider then the class of the groups with all 3-maximal subgroups subnormal [23].

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The research of the second author is supported by a NNSF grant of China (*Grant # 11101369*) *and the Science Foundation of Zhejiang Sci-Tech University under grant 1013843-Y.*

Поступила в редакцию 02.05.14.